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NONNEUTRAL PLASMA EXPANSION

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ABSTRACT

The self similar expansion of nonneutral plasmas is examined using various models. Analytical solutions are obtained for: (1) The expansion of a Maxwellian electron cloud governed by Ohm's Law; (2) The expansion of an electron cloud using a mobility model; (3) The expansion of an electron cloud using a cold plasma model; and (4) The expansion of a charge particle cloud with a temporally decaying nonlinear diffusion coefficient.

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NONNEUTRAL PLASMA EXPANSION

K. E. Lonngren*

In a previous report,[1] we presented a methodical procedure for obtaining the self similar variables using some of the techniques of Lie Group Theory. General physical phenomena which occur in plasmas and could be modeled with equations amenable to a self-similar treatment were presented along with their solutions. An extensive list of references was also given. The purpose of this report is to summarize a pot-pourri of further examples which fall under the umbrage of "Nonneutral Plasma Expansion".

With the increased interest in relativistic electron beam fusion devices, considerable attention has been given to the study of nonneutral plasmas. The recent monograph by Davidson[2] summarizes the past work and motivates our interest in examining various aspects of this problem. In this report, we shall examine three sets of fluid equation models and using the technique of "Self Similar Solution of Partial Differential Equations, obtain analytical solutions.

In addition, a recent experiment on the Wisconsin Multipole suggests that the cross-field diffusion coefficient can be modeled with a one-dimensional diffusion equation where the diffusion coefficient is nonlinear and exponentially decaying in time.[3] A self-similar solution of this problem shall also be given.

In Section II, we describe the various models and present the self-similar solutions. The models are: (1) The expansion of a Maxwellian electron cloud governed by Ohm's Law; (2) The expansion of an electron cloud using a mobility model; (3) The expansion of an electron cloud using a cold plasma model; and (4) The expansion of a charge particle cloud with a temporally decaying nonlinear diffusion coefficient. Section III is the conclusion.

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II. Various Problems

To analyze the problems, we follow the procedure given in reference 1 and only describe the physical phenomena, list the PDE, the self similar variables, the ODE and the solution to the ODE without repeating the details of the procedure in each case.

1) Expansion of a Maxwellian electron cloud governed by Ohm's Law[4]

In this example, we examine the expansion into a vacuum of a thermalized electron cloud, described by an isothermal Maxwellian distribution. It is assumed that collisions are sufficiently frequent such that we can speak of a conductivity for the medium.

The governing equations are:

I) Equation of continuity

$$j_x + \rho_t = 0 \tag{1}$$

II) Ohm's Law

$$j = -\sigma\Phi_x \quad \text{and} \tag{2}$$

III) Maxwellian electrons

$$\rho = n_0 q \exp[q\Phi/k_B T_e] \tag{3}$$

where all symbols are standard. By differentiating (3) with respect to x and substituting (2), we obtain

$$j = -\sigma \frac{k_B T_e}{q} \frac{1}{\rho} \rho_x \tag{4}$$

In normalized units, (4) and (1) lead to

$$\left(\frac{1}{\psi} \psi_y\right)_y - \psi_\tau = 0 \tag{5}$$

where

$$\psi = \rho/n_0 q, \quad y = x/\lambda_D, \quad \tau = t/(\varepsilon_0/\sigma)$$

and λ_D is the Debye length and ε_0/σ is a relaxation time.

We find that (5) admits a self-similar solution of the form

$$\eta(\xi) = \tau\psi(y,\tau), \quad \xi = y/\tau . \tag{6}$$

The solution satisfies the conservation law that electron charge is conserved in space.

The boundary conditions which are germane to this problem are:

a) $\psi(0,\tau) = \psi_0/\tau \Rightarrow \eta(\xi=0) = \psi_0$

b) $\psi(\infty,\tau) = 0 = \psi(y,0) \Rightarrow \eta(\xi=\infty) = 0$ i.e. "consolidation". (7)

Substituting (6) into (5), we obtain

$$[\frac{1}{\eta}\eta_\xi]_\xi = -[\xi\eta]_\xi \quad . \tag{8}$$

The first integral of (8) is

$$-\xi\eta + C_1 = \frac{1}{\eta}\eta_\xi \quad . \tag{9}$$

We shall further impose the condition that the current $j \to 0$ as $y \to \infty$. From (4), this transforms to $\frac{1}{\eta}\eta_\xi \to 0$ as $\xi \to \infty$. Using this and (7b), $C_1 = 0$.

The integral of (9) is

$$\eta = \frac{1}{\xi^2/2 + C_2} \tag{10}$$

The constant C_2 is determined from (7a) to be $C_2 = 1/\psi_0$. In terms of y and τ, the final result is

$$\psi = \frac{\rho}{n_0 q} = \frac{1}{\tau[y^2/2\tau^2 + 1/\psi_0]} \quad . \tag{11}$$

Charge is conserved as is shown below.

$$\int_0^\infty \frac{\rho}{n_0 q}\,dy = \frac{1}{n_0 q}\int_0^\infty \frac{d\xi}{[\xi^2/2 + 1/\psi_0]} = \frac{\pi}{n_0 q}\sqrt{\frac{\psi_0}{2}} = \text{const.} \tag{12}$$

In conclusion, we have examined the expansion of an electron cloud in a vacuum. Under conditions where Coulomb forces can be neglected, this calculation could model an electron cloud expansion in a plasma in time scales short with respect to ion motion.

2) Expansion of an electron cloud using a mobility model[5]

Recently, considerable attention has been given to the problem of the transient behavior of the bulk electric field and space charge distribution in semi-conductors and in the conduction in dielectric and insulating fluids. It has been found prudent to use a

mobility model where the velocities of charge carriers injected from an emitting electrode are proportional to the electric field through their mobilities and the electric field is related to the charge densities of the carriers through Gauss's law. Many and Rakavy [6] and Helfrich and Mark [7] were probably the first to suggest that the problem could be modeled with the set of dimensionless equations

$$
\begin{aligned}
&E_x = \rho \\
&i_x + \rho_t = 0 \qquad\qquad (13) \\
&i = \rho E
\end{aligned}
$$

which are Poisson's equation, the equation of continuity and a mobility definition for current respectively. The subscript x and t denote a partial differentiation with respect to space and time.

In their original paper, Many and Rakavy [6] obtained a solution to (13) by looking for the characteristics of the problem. Subsequently, this approach was extended by Batra, Schechtman and Seki, [8] Zahn, Tsang and Pao, [9] de Oliveira and Ferreira [10] and others. An extensive list of relevant experimental observations is given in reference 9.

As the problem is extremely important, we suggest an alternative technique for solution which will describe the spatial and temporal evolution of: I) a fixed electric field and II) a constant source of current which are both governed by (13). The technique that we shall apply is to find the "self-similar solution" of this set of partial differential equations.

Equation (13) can be written as

$$EE_{xx} + (E_x)^2 + E_{xt} = 0 \ . \qquad (14)$$

The self similar variables are

$$\phi = \frac{E}{t^{\alpha/\gamma}} \quad \text{and} \quad \xi = \frac{x}{t^{\beta/\gamma}} \qquad (15)$$

where α/γ and β/γ are constants which will be specified by invariance and conservation requirements. Invariance specifies that $\alpha - \beta = -\gamma$.

Substituting (15) in (14), we write

$$(\phi - \frac{\beta}{\gamma}\xi)\phi_{\xi\xi} + (\frac{\alpha}{\gamma} - \frac{\beta}{\gamma})\phi_{\xi} + (\phi_{\xi})^2 = 0 \quad . \tag{16}$$

We shall obtain solutions for (16) subject to: I) A fixed electric field and II) a constant current at $x = 0$ requirement.

I) Electric field is constant at $x = 0$. We choose:

$$\alpha = 0$$

$$\beta = \gamma \, .$$

With these values, $\xi = x/t$ and $\phi = E$. The integral of (16) with these constants is

$$\phi\phi_{\xi} - \xi\phi_{\xi} = k_1 \tag{17}$$

where k_1 is a constant of integration. The constant is set equal to zero since $E(x = 0,t) = 0$ in order to satisfy space charge limited conditions. This specifies $\phi(\xi = 0) = 0$. The solution of (17) is $\phi = \xi$ from which we compute that

$$E = \frac{x+x_0}{t+t_0}$$

$$\rho = \frac{1}{t+t_0} \tag{18}$$

$$i = \frac{(x+x_0)}{(t+t_0)^2}$$

where the constants x_0 and t_0 have been explicitely included since (13) is invariant to translation.

II) Current is constant at $x = 0$. We choose:

$$\frac{2\alpha}{\gamma} = \frac{\beta}{\gamma} \, .$$

With this choice, we have $\xi = \frac{x}{t^2}$ and $\phi = \frac{E}{t}$ where ϕ satisfies (16) which becomes

$$(\phi - 2\xi)\phi_{\xi\xi} - \phi_{\xi} + (\phi_{\xi})^2 = 0 \, . \tag{20}$$

This can be integrated once to

$$(\phi - 2\xi)\phi_\xi + \phi = k_2 . \tag{21}$$

The constant of integration k_2 is set equal to zero since we require that $E(x = 0,t)$ be zero for space charge limited conditions which specifies $\phi(\xi = 0) = 0$. The integral of (21) is

$$\phi = \xi \tag{22}$$

from which we compute that

$$E = \frac{x+x_0}{t+t_0}$$

$$\rho = \frac{1}{t+t_0} \tag{23}$$

$$i = \frac{x+x_0}{(t+t_0)^2}$$

where again the constants x_0 and t_0 have been introduced since (13) is invariant to translation. Note that this is identical to (18).

In (21), we can also obtain the solution for a non-space charge limited condition $(E(x = 0,t) \neq 0)$ by setting the constant of integration k_2 equal to, say, $2i_0$. The integral of (21) can then be written as

$$\phi = \xi + i_0 \tag{24}$$

from which we compute that

$$E = (t + t_0)(i_0 + \frac{x+x_0}{(t+t_0)^2})$$

$$\rho = \frac{1}{t+t_0} \tag{25}$$

$$i = \frac{x+x_0}{(t+t_0)^2} + i_0$$

where again the constants x_0 and t_0 have been reintroduced. Note that (25) reduces to the space charge limited case for $i_0 = 0$.

In conclusion, we have shown that the set of equations which describe the Transient Space Charge Limited Current Problem admit self-similar solutions for two physically interesting boundary conditions. These solutions are valid in the initial stages before

the particles reach a second electrode which may be placed at $x = L$.

3. Expansion of an electron cloud using a cold plasma model

A model which can describe the behavior of an electron cloud expansion into a fixed ion background is to assume that Poisson's equation is an initial condition. The electrostatic approximation for the $\nabla \times \overline{B}$ Maxwell equation and the continuity equation assure that Poisson's equation is satisfied for all time.[11]

The basic equations are:

$$
\begin{aligned}
(\rho v)_x + \rho_t &= 0 \\
mv_t + mvv_x &= -eE \\
\nabla \times \overline{B} \sim 0 &= \varepsilon_0 E_t - \rho v
\end{aligned}
\qquad (26)
$$

which are the equations of continuity and motion and Maxwell's equation respectively.

The self similar variables which satisfy the conservation law that $\int_0^\infty ndx$ = constant are

$$\varepsilon = E;\ N = nt^2;\ u = v/t;\ \xi = x/t^2 \ . \qquad (27)$$

These are the same self similar variables that were obtained in an earlier study of the set (26) where the Ansatz that Maxwell's equation could replace Poisson's equation had not been made.[12] In the earlier study, it was not a pedestrian task to integrate the ODE.

Substituting (27) into (26), we now obtain the ODE:

$$
\begin{aligned}
-2N - 2\xi N_\xi + (NU)_\xi &= 0 \\
U - 2\xi U_\xi + UU_\xi &= -\varepsilon \\
2\xi \varepsilon_\xi + NU = 0 \ .
\end{aligned}
\qquad (28)
$$

A solution of this set is

$$U = 2\xi \ , \quad N = 2 \quad \text{and} \quad \varepsilon = -2\xi \qquad (29)$$

from which we write the solution of (26) using (27) and (29) as

$$E = -\frac{2x}{t^2} \quad n = \frac{2}{t^2} \quad v = \frac{2x}{t} \ . \qquad (30)$$

In conclusion, we find that in the final self similar solution, the density is

independent of position at the end rather than making it an a priori assumption in the calculation as did Gintsburg who treated a similar problem.[13]

4. Expansion of charged particles with a temporally decaying nonlinear diffusion coefficient

In recent experiments on the Wisconsin Multipole, it was confirmed that the cross-field diffusion coefficient depended on time and amplitude as[3]

$$D \sim \frac{\varepsilon^{-\alpha t}}{\sqrt{n}} \quad . \tag{31}$$

Incorporating this in the one dimensional diffusion equation, we obtain

$$n_t = \left[\varepsilon^{-\alpha t} \frac{1}{\sqrt{n}} n_x\right]_x \tag{32}$$

where all constants except α have been suitably normalized away. A change of variables

$$\tau = \frac{1}{\alpha} [1 - \varepsilon^{-\alpha t}] \tag{33}$$

transforms (32) to

$$n_\tau = \left[\frac{1}{\sqrt{n}} n_x\right]_x \quad . \tag{34}$$

The self similar treatment of (34) is straightforward, at least for the case where the conservation law $\int_0^\infty n dx = \text{constant}$ is valid. The self similar variables are

$$N = n\tau^{2/3} \quad \text{and} \quad \xi = x/\tau^{2/3} \tag{35}$$

and the resulting ODE is

$$-\frac{2}{3}(\xi N)_\xi = \left[\frac{1}{\sqrt{N}} N_\xi\right]_\xi \quad . \tag{36}$$

If the burst of particles is symmetric at $x = 0$ such that $n_x\big|_{x=0} = 0$, and $n(x=0,\tau) = \tau^{-2/3}$ (36) can be integrated twice to yield

$$\sqrt{N} = \frac{2}{2 + \xi^2/3} \quad . \tag{37}$$

Using (33) and (35) in (37), we finally obtain

$$n(x,t) = \frac{4[\frac{1}{\alpha}(1-\varepsilon^{-\alpha t})]^2}{[2(\frac{1}{\alpha}(1-\varepsilon^{-\alpha t}))^{4/3} + \frac{x^2}{3}]^2} . \tag{38}$$

We note that (38) gives a reasonably accurate qualitative description to the experimental results. This seems true even though the time scale in the experiment is sufficiently long such that normal modes have been exited.

III. Conclusion

The self similar behavior of four plasma phenomena have been described.

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